Consider the curve $\vec{r}(t) = < \sin t - t\cos t, 1 - t^2, \cos t + t\sin t >$.

SCORE: _____ / 60 PTS

[a] Find the unit tangent vector $\vec{T}(t)$.

6

$\vec{r}\,'(t) = \underline{\langle t\sin t, -2t, t\cos t\rangle} = t\langle \sin t, -2, \cos t\rangle$

$\|\vec{r}\,'(t)\| = |t|\sqrt{\sin^2 t + 4 + \cos^2 t} = \underline{\sqrt{5}|t|}$, 6

$\vec{T}(t) = \frac{1}{\sqrt{5}|t|} t\langle \sin t, -2, \cos t\rangle$

$= \begin{cases} -\frac{1}{\sqrt{5}}\langle \sin t, -2, \cos t\rangle, & t < 0 \\ \frac{1}{\sqrt{5}}\langle \sin t, -2, \cos t\rangle, & t > 0 \end{cases}$

6

IF YOU ASSUMED THAT $t > 0$
AND ONLY GOT $\frac{1}{\sqrt{5}}\langle \sin t, -2, \cos t\rangle$,
YOU WILL GET FULL CREDIT

[b] Find parametric equations for the tangent line to the curve at the point $(\pi, 1 - \pi^2, -1)$.

$1 - t^2 = 1 - \pi^2 \longrightarrow t = \pi, -\pi$

$\downarrow \quad \downarrow$

$x = \pi \quad \cancel{x = -\pi}$

$\vec{r}\,'(\underline{\pi}) = \underline{\pi\langle 0, -2, -1\rangle}$, 2

4 USE $\underline{\langle 0, 2, 1\rangle}$, 2

4 $\begin{cases} x = \pi \\ y = 1 - \pi^2 + 2t \\ z = -1 + t \end{cases}$

[c] Find the curvature function $\kappa(t)$.

$\vec{T}\,'(t) = \begin{cases} -\frac{1}{\sqrt{5}}\langle \cos t, 0, -\sin t\rangle, & t < 0 \\ \frac{1}{\sqrt{5}}\langle \cos t, 0, -\sin t\rangle & t > 0 \end{cases}$ 7

$\kappa(t) = \frac{\|\vec{T}\,'(t)\|}{\|\vec{r}\,'(t)\|} = \frac{\frac{1}{\sqrt{5}}\sqrt{\cos^2 t + \sin^2 t}}{\sqrt{5}|t|} = \underline{\frac{1}{5|t|}}$ 5 3 3

[d] Find the length of the curve from the point $(\pi, 1 - \pi^2, -1)$ to the point $(-2\pi, 1 - 4\pi^2, 1)$.

$1 - t^2 = 1 - 4\pi^2 \longrightarrow t = 2\pi, -2\pi$

$\swarrow \quad \downarrow$

$x = -2\pi \quad \cancel{x = 2\pi}$

4

$\int_\pi^{2\pi} \underline{\|\vec{r}\,'(t)\|}\,dt = \int_\pi^{2\pi} \underline{\sqrt{5}t}\,dt = \underline{\frac{\sqrt{5}}{2}t^2}\Big|_\pi^{2\pi}$

2 2 2

$= \frac{\sqrt{5}}{2}(4\pi^2 - \pi^2) = \underline{\frac{3\sqrt{5}}{2}\pi^2}$ 2

Find the curvature of $\vec{r}(t) = te^t\,\vec{i} - 2\cos t\,\vec{j} + te^{-t}\,\vec{k}$ at the point $(0, -2, 0)$.

$te^t = 0 \rightarrow t = 0$

SCORE: _____ / 20 PTS

$\vec{r}'(t) = \langle e^t + te^t, 2\sin t, e^{-t} - te^{-t} \rangle$ (4) $\quad \vec{r}'(0) = \langle 1, 0, 1 \rangle$ (1)

$\vec{r}''(t) = \langle 2e^t + te^t, 2\cos t, -2e^{-t} + te^{-t} \rangle$ (4) $\quad \vec{r}''(0) = \langle 2, 2, -2 \rangle$ (1)

$$\kappa(0) = \frac{\|\langle 1, 0, 1\rangle \times \langle 2, 2, -2\rangle\|}{\|\langle 1, 0, 1\rangle\|^3} = \frac{\|\langle -2, 4, 2\rangle\|}{\sqrt{2}^3} \quad (4)\ (3)$$

$$= \frac{\cancel{2}\|\langle -1, 2, 1\rangle\|}{\cancel{2}\sqrt{2}}$$

$$= \frac{\sqrt{6}}{\sqrt{2}}$$

$$= \sqrt{3} \quad (3)$$

Find a vector function for the curve of intersection of the surfaces $x^2 + y^2 + z^2 = 34$ and $x - z = 6$. **SCORE: _____ / 25 PTS**

$$x = z + 6$$

$$\underline{(z+6)^2 + y^2 + z^2 = 34} \quad 5$$

$$2z^2 + 12z + y^2 = -2$$

$$2(z^2 + 6z + 9) + y^2 = -2 + 18 = 16$$

$$2(z+3)^2 + y^2 = 16$$

$$\underline{\frac{(z+3)^2}{8} + \frac{y^2}{16} = 1} \quad 8$$

$$\frac{(z+3)^2}{8} = \cos^2 t \qquad \frac{y^2}{16} = \sin^2 t$$

$$3 \quad \underline{z = -3 + 2\sqrt{2}\cos t} \qquad \underline{y = 4\sin t} \quad 3$$

$$3 \quad \underline{x = 3 + 2\sqrt{2}\cos t}$$

$$\underline{\vec{r}(t) = \langle 3 + 2\sqrt{2}\cos t,\ 4\sin t,\ -3 + 2\sqrt{2}\cos t \rangle} \quad 3$$

Consider the graph of $y = 3\ln x$. **SCORE: _____ / 20 PTS**

[a] Find the curvature at the point $(4, 3\ln 4)$.

$f'(x) = 3x^{-1}$ $\quad f'(4) = \frac{3}{4}$

$f''(x) = -3x^{-2}$ $\quad f''(4) = -\frac{3}{16}$

$$K(4) = \frac{\left|-\frac{3}{16}\right|}{\sqrt{1+\left(\frac{3}{4}\right)^2}^{\,3}} = \frac{\frac{3}{16}}{\left(\frac{5}{4}\right)^3} = \frac{3}{\cancel{16}} \cdot \frac{4^{\cancel{3}}}{5^3} = \frac{12}{125}$$

7 7 6

[b] **BONUS QUESTION (5 POINTS):**

Suppose $y = f(x)$ is a function which is infinitely differentiable (ie. all derivatives exist at all points in the domain).

Explain why the inflection points of f are points of minimum curvature. Justify your answer algebraically.

$K \geq 0$ (3) f'' @ INFLECTION POINT $= 0$ SINCE f'' EXISTS EVERYWHERE

(1) SO $K = 0$ @ INFLECTION POINT IE. MINIMUM K

(1)

Let $\vec{u}(t) = \vec{r}(t) \cdot [\vec{r}\,'(t) \times \vec{w}(t)]$. **SCORE: _____ / 25 PTS**

[a] Find an expression for $\vec{u}\,'(t)$.

<u>**NOTE: Your final answer must be completely simplified.**</u>

<u>**It may use the derivatives of vector functions, but it must NOT use the derivatives of dot and cross products.**</u>

$\vec{u}\,'(t) = \vec{r}\,'(t) \cdot [\vec{r}\,'(t) \times \vec{w}(t)] + \vec{r}(t) \cdot [\vec{r}\,''(t) \times \vec{w}(t) + \vec{r}\,'(t) \times \vec{w}\,'(t)]$ 5

5

BUT $\vec{r}\,'(t) \cdot [\vec{r}\,'(t) \times \vec{w}(t)] = 0$ SINCE CROSS PRODUCT IS ORTHOGONAL TO ORIGINAL VECTORS

5

SO $\vec{u}\,'(t) = \vec{r}(t) \cdot [\vec{r}\,''(t) \times \vec{w}(t) + \vec{r}\,'(t) \times \vec{w}\,'(t)]$

3

[b] In addition, suppose $\vec{w}(t)$ is a function such that $\vec{w}(t)$ and $\vec{r}\,''(t)$ always point in exact opposite directions.

Find an expression for $\vec{u}\,'(t)$.

<u>**NOTE: Your final answer must be completely simplified.**</u>

$\|\vec{r}\,''(t) \times \vec{w}(t)\| = \|\vec{r}\,''(t)\| \, \|\vec{w}(t)\| \sin 180^\circ = 0$

SO $\vec{r}\,''(t) \times \vec{w}(t) = \vec{0}$ 6

SO $\vec{u}\,'(t) = \vec{r}(t) \cdot [\vec{r}\,'(t) \times \vec{w}\,'(t)]$ 2